

Accuracy of the pion elastic form factor extracted from a local-duality sum rule

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We analyze the accuracy of the pion elastic form factor predicted by a local-duality (LD) version of dispersive sum rules. To probe the precision of this theoretical approach, we adopt potential models with interactions that involve both Coulomb and confining terms. In this case, the exact form factor may be obtained from the solution of the Schrödinger equation and confronted with the LD sum-rule results. We use parameter values appropriate for hadron physics and observe that, independently of the details of the confining interaction, the deviation of the LD form factor from the exact form factor culminates in the region $Q^2 \approx 4\text{--}6 \text{ GeV}^2$. For larger Q^2 , the accuracy of the LD description increases rather fast with Q^2 . A similar picture is expected for QCD. For the pion form factor, existing data suggest that the LD limit may be reached already at the relatively low values $Q^2 = 4\text{--}10 \text{ GeV}^2$. Thus, large deviations of the pion form factor from the behaviour predicted by LD QCD sum rules for higher values of Q^2 , as found by some recent analyses, appear to us quite improbable. New accurate data on the pion form factor at $Q^2 = 4\text{--}10 \text{ GeV}^2$ expected soon from JLab will have important implications for the behaviour of the pion form factor in a broad Q^2 range up to asymptotically large values of Q^2 .

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1. INTRODUCTION

In spite of a rather long history of theoretical studies of the pion elastic form factor, no consensus on its behaviour in the spacelike region $Q^2 \geq 2\text{--}4 \text{ GeV}^2$ has been reached so far. For instance, recent theoretical investigations [1–3] report results for the pion form factor much larger than earlier ones (cf. Fig. 1): According to [1–3], even at $Q^2 \approx 50\text{--}100 \text{ GeV}^2$ the pion form factor $F_\pi(Q^2)$ remains much larger than the asymptotic behaviour expected from perturbative QCD [8]

$$Q^2 F_\pi(Q^2) = 8\pi\alpha_s(Q^2)f_\pi^2, \quad (1.1)$$

with f_π the decay constant of the pion and $\alpha_s(Q^2)$ the strong coupling. Subleading logarithmic and power corrections modify the behaviour (1.1) at large but finite Q^2 . In early applications of QCD one hoped that power corrections would vanish fast enough with Q^2 ; however, later investigations revealed that nonperturbative power corrections dominate the form factor $F_\pi(Q^2)$ up to relatively high $Q^2 \approx 10\text{--}20 \text{ GeV}^2$. This picture has arisen from different approaches [9–14]. At this stage, the conclusion was that even for Q^2 as large as $Q^2 = 20 \text{ GeV}^2$ the $O(1)$ term provides about half of the form factor and the perturbative-QCD formula based on factorization starts to work well only at $Q^2 \geq 50\text{--}100 \text{ GeV}^2$.

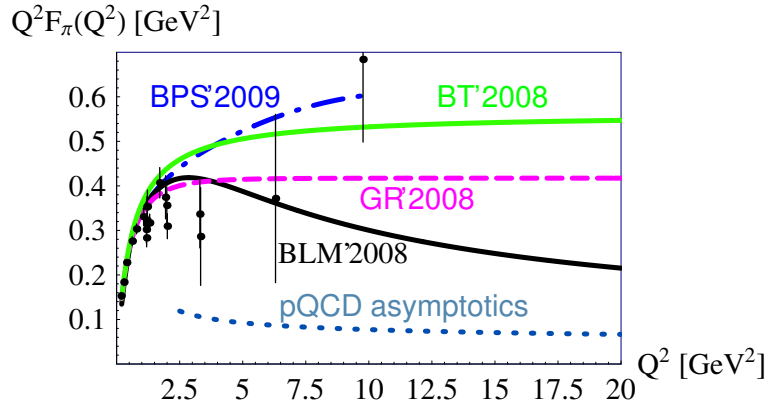


Fig. 1: Some predictions for the pion elastic form factor $F_\pi(Q^2)$ — lower solid (black) line: BLM'2008 [4], upper solid (green) line: BT'2008 [1], dashed (magenta) line: GR'2008 [2], and dash-dotted (blue) line: BPS'2009 [5, Eq. (4.11b)] — vs. experiment [6, 7].

A convincing and largely model-independent argument comes from QCD sum rules in their local-duality (LD) version [15], which enables one to take into account, on an equal footing, the $O(1)$ and the $O(\alpha_s)$ contributions to $F_\pi(Q^2)$: the LD form of QCD sum rules predicts in an essentially model-independent way the relative weights of these contributions [4], but needs an additional input — the Q^2 -dependent effective threshold $s_{\text{eff}}(Q^2)$ — in order to calculate $F_\pi(Q^2)$. All previous applications of LD sum rules had to rely on assumptions about the Q^2 behaviour of the effective threshold. The LD *model* [15, 16] assumes that reasonable estimates for the form factor already starting from $Q^2 \geq 2\text{--}4 \text{ GeV}^2$ may be obtained by setting $s_{\text{eff}}(Q^2)$ equal to its LD value $s_{\text{LD}} = 4\pi^2 f_\pi^2$. Under this assumption, $F_\pi(Q^2)$ has been calculated in [4].¹

Surprisingly, recent studies [1–3, 5] find large deviations from the LD results [4]. To quantify these deviations, Fig. 2 compares the equivalent effective thresholds² computed from the results of these analyses. For some of these studies, the deviation of the equivalent threshold from s_{LD} rises with Q^2 , although the accuracy of the LD model is expected to increase with Q^2 . However, all of these studies involve explicit or implicit assumptions.

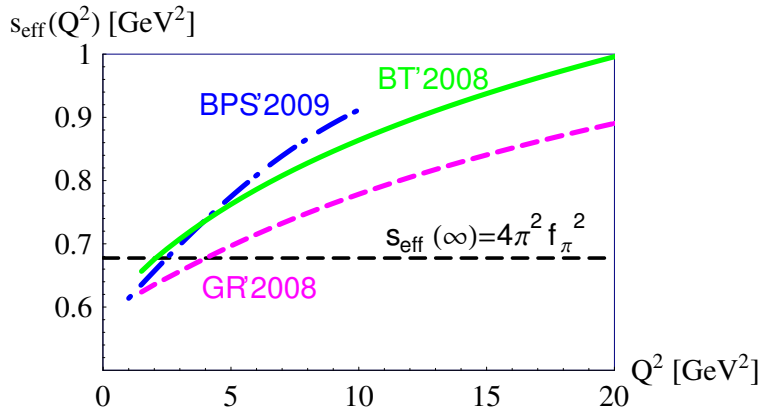


Fig. 2: Equivalent effective thresholds $s_{\text{eff}}(Q^2)$ resulting from the findings of [1, 2, 5], identified by the same line codes as in Fig. 1.

The goal of this paper is to test the accuracy of the LD model by exploiting the fact that in quantum mechanics the form factor may be found in two ways, *viz.*, by the LD sum rule and by solving the Schrödinger equation. For the first time we calculate the exact Q^2 -dependent threshold in a quantum-mechanical potential model for different confining interactions. We find that the accuracy of the LD approximation at a given value of Q^2 depends on the details of the confining interaction. More importantly, we observe a universal feature that does not depend on these details: namely, for realistic values of the parameters of the potential model relevant for hadron physics, the accuracy of the LD model increases with Q^2 already starting at relatively low values $Q^2 \approx 4\text{--}6 \text{ GeV}^2$.

In QCD, experimental data on the pion form factor at small Q^2 indicate that the LD limit may be reached already at relatively low values of $Q^2 \approx 5\text{--}6 \text{ GeV}^2$. This, of course, does not mean that the asymptotic pQCD formula describes well the form factor in this region: the $O(1)$ term still dominates the form factor up to $Q^2 \approx 20 \text{ GeV}^2$. Therefore, we conclude that large and growing-with- Q^2 deviations from the LD limit in the region $Q^2 \approx 20\text{--}50 \text{ GeV}^2$ implied by the recent analyses [1–3] seem to us very unlikely.

This paper is organized as follows: In the next section, after recalling rather well-known basics of the LD limit of the QCD sum-rule approach we formulate, for use in these LD sum rules, a model for $s_{\text{eff}}(Q^2)$ slightly more sophisticated than the one used in [4]. In Sec. 3, we discuss in detail rigorous features and assumptions employed in the LD approach. Section 4 presents the quantum-mechanical counterpart of the LD model for $F_\pi(Q^2)$ using potentials that involve both Coulombic and confining interactions: we examine the impact of the specific shape of the confining interactions on the accuracy of the LD model for the elastic form factor. Section 5 summarizes our conclusions and outlook. Appendix A presents several technical details of perturbative two-loop calculations in quantum mechanics.

¹ Notice that the pioneering work on the LD model [15] used a simple interpolation formula for the unknown $O(\alpha_s)$ contribution to $F_\pi(Q^2)$. The first analysis of the pion form factor taking into account the $O(\alpha_s)$ contribution obtained in [17] was performed in [4].

² The *equivalent effective threshold* is defined by requiring that a given theoretical prediction for the pion form factor is reproduced by the LD sum rule (2.9) if the corresponding equivalent threshold is used (cf. Sect. 2).

2. SUM RULE

The fundamental objects for a sum-rule extraction of pion features are the two- and three-point correlation functions

$$\begin{aligned}\Pi(p^2) &= \int d^4x e^{ipx} \langle \Omega | T[j(x) j^\dagger(0)] | \Omega \rangle, \\ \Gamma(p_1^2, p_2^2, q^2) &= \int d^4x_1 d^4x_2 e^{i(p_1x_1 - p_2x_2)} \langle \Omega | T[j(x_1) J(0) j^\dagger(x_2)] | \Omega \rangle, \quad q \equiv p_1 - p_2, \quad Q^2 \equiv -q^2;\end{aligned}\quad (2.1)$$

where Ω labels the physical vacuum, j is shorthand for the interpolating axial current $j_{5\alpha}$ of the positively charged pion, $\langle \Omega | j_{5\alpha}(0) | \pi(p) \rangle = ip_\alpha f_\pi$, J labels the electromagnetic current J_μ , and for brevity we omit all Lorentz indices. In QCD, the correlators (2.1) can be found by applying their OPEs. Instead of discussing the Green functions (2.1) in Minkowski space, it is convenient to study the time-evolution operators in Euclidean space, which arise upon performing the Borel transformation $p^2 \rightarrow \tau$ to a parameter τ related to Euclidean time. The Borel image of the two-point correlator $\Pi(p^2)$ is

$$\Pi_{\text{OPE}}(\tau) = \int_0^\infty ds e^{-s\tau} \rho_{\text{pert}}(s) + \Pi_{\text{cond}}(\tau), \quad \rho_{\text{pert}}(s) = \rho_0(s) + \alpha_s \rho_1(s) + O(\alpha_s^2), \quad (2.2)$$

with spectral densities $\rho_i(s)$ related to perturbative two-point graphs, and nonperturbative power corrections $\Pi_{\text{cond}}(\tau)$. At hadron level, insertion of intermediate hadron states casts the Borel-transformed two-point correlator into the form

$$\Pi(\tau) = f_\pi^2 e^{-m_\pi^2 \tau} + \text{excited states}. \quad (2.3)$$

In this expression for $\Pi(\tau)$, the first term on the right-hand side constitutes the pion contribution. Applying the double Borel transform $p_{1,2}^2 \rightarrow \tau/2$ to the three-point correlator $\Gamma(p_1^2, p_2^2, q^2)$ results, at QCD level, in

$$\begin{aligned}\Gamma_{\text{OPE}}(\tau, Q^2) &= \int_0^\infty \int_0^\infty ds_1 ds_2 \exp\left(-\frac{s_1 + s_2}{2} \tau\right) \Delta_{\text{pert}}(s_1, s_2, Q^2) + \Gamma_{\text{cond}}(\tau, Q^2), \\ \Delta_{\text{pert}}(s_1, s_2, Q^2) &= \Delta_0(s_1, s_2, Q^2) + \alpha_s \Delta_1(s_1, s_2, Q^2) + O(\alpha_s^2),\end{aligned}\quad (2.4)$$

where $\Delta_{\text{pert}}(s_1, s_2, Q^2)$ is the double spectral density of the three-point graphs of perturbation theory and $\Gamma_{\text{cond}}(\tau, Q^2)$ labels the power corrections. Inserting intermediate hadron states yields, for the hadron-level expression for $\Gamma(\tau, Q^2)$,

$$\Gamma(\tau, Q^2) = F_\pi(Q^2) f_\pi^2 e^{-m_\pi^2 \tau} + \text{excited states}. \quad (2.5)$$

Quark-hadron duality assumes that above effective continuum thresholds s_{eff} the excited-state contributions are dual to the high-energy regions of the perturbative graphs. In this case, the relevant sum rules read in the chiral limit [15, 20]

$$\begin{aligned}f_\pi^2 &= \int_0^{\bar{s}_{\text{eff}}(\tau)} ds e^{-s\tau} \rho_{\text{pert}}(s) + \frac{\langle \alpha_s G^2 \rangle}{12\pi} \tau + \frac{176\pi \alpha_s \langle \bar{q}q \rangle^2}{81} \tau^2 + \dots, \\ F_\pi(Q^2) f_\pi^2 &= \int_0^{s_{\text{eff}}(Q^2, \tau)} \int_0^{s_{\text{eff}}(Q^2, \tau)} ds_1 ds_2 \Delta_{\text{pert}}(s_1, s_2, Q^2) \exp\left(-\frac{s_1 + s_2}{2} \tau\right) \\ &\quad + \frac{\langle \alpha_s G^2 \rangle}{24\pi} \tau + \frac{4\pi \alpha_s \langle \bar{q}q \rangle^2}{81} \tau^2 (13 + Q^2 \tau) + \dots.\end{aligned}\quad (2.6)$$

As a consequence of the use of local condensates, the right-hand side of (2.7) involves polynomials in Q^2 and therefore increases with Q^2 , whereas the form factor $F_\pi(Q^2)$ on the left-hand side should decrease with Q^2 . Therefore, at large Q^2 the sum rule (2.7), with its truncated series of power corrections, cannot be directly used. There are essentially only two ways for considering the region of large Q^2 .

One remedy is the resummation of all power corrections: the resummed power corrections decrease with increasing Q^2 . This may be achieved by the introduction of nonlocal condensates [18] in a, however, model-dependent manner [19].

Another — rather simple — option is to fix the Borel parameter τ to the value $\tau = 0$, thus arriving at a *local-duality sum rule* [15]. Therein all power corrections vanish and the remaining perturbative term decreases with Q^2 . In the LD

limit, one finds

$$f_\pi^2 = \int_0^{\bar{s}_{\text{eff}}} ds \rho_{\text{pert}}(s) = \frac{\bar{s}_{\text{eff}}}{4\pi^2} \left(1 + \frac{\alpha_s}{\pi}\right) + O(\alpha_s^2), \quad (2.8)$$

$$F_\pi(Q^2) f_\pi^2 = \int_0^{s_{\text{eff}}(Q^2)} \int_0^{s_{\text{eff}}(Q^2)} ds_1 ds_2 \Delta_{\text{pert}}(s_1, s_2, Q^2). \quad (2.9)$$

The spectral densities $\rho_{\text{pert}}(s)$ and $\Delta_{\text{pert}}(s_1, s_2, Q^2)$ are calculable by perturbation theory. Hence, by fixing \bar{s}_{eff} and $s_{\text{eff}}(Q^2)$, it is straightforward to extract the pion's decay constant f_π and form factor $F_\pi(Q^2)$.

Noteworthy, the effective and the physical thresholds are different quantities: The latter is a constant determined by the masses of the hadron states. The effective thresholds \bar{s}_{eff} and s_{eff} are parameters of the sum-rule method related to the specific realization of quark–hadron duality; in general, they are *not* constant but depend on external kinematical variables [21, 22].

Let us recall the important properties of the spectral densities on the right-hand sides of (2.2) and (2.4): For $Q^2 \rightarrow 0$, the Ward identity relates the spectral densities $\rho_i(s)$ and $\Delta_i(s_1, s_2, Q^2)$ of two- and three-point functions to each other:

$$\lim_{Q^2 \rightarrow 0} \Delta_i(s_1, s_2, Q^2) = \rho_i(s_1) \delta(s_1 - s_2), \quad i = 0, 1, \dots \quad (2.10)$$

For $Q^2 \rightarrow \infty$ and $s_{1,2}$ kept fixed, explicit calculations [17] yield

$$\lim_{Q^2 \rightarrow \infty} \Delta_0(s_1, s_2, Q^2) \propto \frac{1}{Q^4}, \quad \lim_{Q^2 \rightarrow \infty} \Delta_1(s_1, s_2, Q^2) = \frac{8\pi}{Q^2} \rho_0(s_1) \rho_0(s_2). \quad (2.11)$$

For the pion form factor $F_\pi(Q^2)$ on the left-hand side of (2.7) two exact features are known, namely, its normalization condition related to current conservation, requiring $F_\pi(0) = 1$, and the factorization theorem (1.1). Obviously, if we set

$$s_{\text{eff}}(Q^2 \rightarrow 0) = \frac{4\pi^2 f_\pi^2}{1 + \alpha_s(0)/\pi}, \quad s_{\text{eff}}(Q^2 \rightarrow \infty) = s_{\text{LD}} \equiv 4\pi^2 f_\pi^2, \quad (2.12)$$

the form factor $F_\pi(Q^2)$ extracted from the LD sum rule (2.9) satisfies both of these rigorous constraints. At small Q^2 , we assume a freezing of $\alpha_s(Q^2)$ at the level 0.3, as is frequently done.

With these preliminaries at our disposal, we are now in the position to formulate our approach to the pion elastic form factor $F_\pi(Q^2)$ within the framework of QCD sum rules in the LD limit:

1. We extract $F_\pi(Q^2)$ from the dispersive three-point QCD sum rule in LD limit (2.7): the latter has the favourable feature that all power corrections vanish and all details of nonperturbative dynamics are encoded in one quantity, the effective threshold $s_{\text{eff}}(Q^2)$. We take into account the perturbative spectral densities up to $O(\alpha_s)$ accuracy.
2. It is easy to construct a model for $s_{\text{eff}}(Q^2)$ by some smooth interpolation between its values at $Q^2 = 0$, defined by the Ward identity, and $Q^2 \rightarrow \infty$, determined by factorization: a simple parametrization with a single constant Q_0^2 fixed by fitting the data at $Q^2 = 1 \text{ GeV}^2$ might read

$$s_{\text{eff}}(Q^2) = \frac{4\pi^2 f_\pi^2}{1 + \alpha_s(0)/\pi} \left[1 + \tanh\left(\frac{Q^2}{Q_0^2}\right) \frac{\alpha_s(0)}{\pi} \right], \quad Q_0^2 = 2.02 \text{ GeV}^2. \quad (2.13)$$

According to Fig. 3, our interpolation perfectly describes the well-measured data in the range $Q^2 \approx 0.5\text{--}2.5 \text{ GeV}^2$.

Note that the effective continuum threshold $s_{\text{eff}}(Q^2)$ in Eq. (2.13) approaches its limit s_{LD} already at $Q^2 \approx 4\text{--}5 \text{ GeV}^2$. For $Q^2 > 4\text{--}5 \text{ GeV}^2$, it practically coincides with the LD effective threshold of [15]. Moreover, for $Q^2 > 5\text{--}6 \text{ GeV}^2$ the formula (2.13) is pretty close to the model of [4].³ Hence, also the resulting prediction for $F_\pi(Q^2)$ is rather close to the one we found earlier [4] (cf. Fig. 1). Obviously, the model labeled BLM in Fig. 1 provides a perfect description of the available $F_\pi(Q^2)$ data in the region $Q^2 = 1\text{--}2.5 \text{ GeV}^2$. For $Q^2 \geq 3\text{--}4 \text{ GeV}^2$, it reproduces well all the data, except for a point at $Q^2 = 10 \text{ GeV}^2$, where it is off the present experimental value, which anyhow has a rather large error, by some two standard deviations.⁴

³ It goes without saying that our present goal is *not* to improve the model of [4] but to probe the accuracy of this model as a function of Q^2 .

⁴ It is virtually inconceivable to construct models compatible with all experimental results within $Q^2 = 2.5\text{--}10 \text{ GeV}^2$, as revealed by closer inspection of Fig. 1: those approaches which hit the data at $Q^2 = 10 \text{ GeV}^2$ overestimate the better-quality data points at $Q^2 \approx 2\text{--}4 \text{ GeV}^2$.

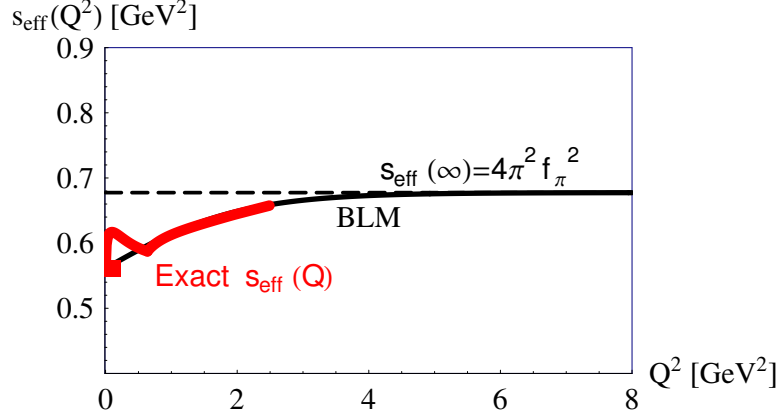


Fig. 3: Effective continuum threshold $s_{\text{eff}}(Q^2)$ as function of Q^2 . Bold (red) line: exact behaviour $s_{\text{eff}}(Q^2)$ as reconstructed from the experimental data for $F_\pi(Q^2)$ [6, 7] for $Q^2 \leq 2.5 \text{ GeV}^2$. Solid (black) line: our simple interpolating model (2.13) for $s_{\text{eff}}(Q^2)$.

3. WHAT IS PREDICTED BY AND WHAT IS CONJECTURED IN THE LD MODEL?

Interestingly, in the region $Q^2 \geq 3\text{--}4 \text{ GeV}^2$ the BLM model yields considerably lower predictions than the results of the different theoretical approaches presented in Refs. [1–3, 5]. For the $F_\pi(Q^2)$ predictions of [1, 2, 5], the corresponding equivalent effective thresholds $s_{\text{eff}}(Q^2)$ recalculated from (2.9) are depicted in Fig. 2: In all instances, they considerably exceed, for larger Q^2 , the LD limit s_{LD} dictated by factorization. Moreover, their deviation from s_{LD} increases with Q^2 .

Let us inspect more carefully what is, in fact, predicted by the LD sum rule and what is conjectured in this approach.

The sum rule (2.9) for the pion form factor relies on two ingredients: first, on the *rigorous calculation* of the spectral densities of the perturbative-QCD diagrams (recall that power corrections vanish in the LD limit $\tau = 0$); second, on the *assumption* of quark–hadron duality, which claims that the contributions of the hadronic continuum states may be well described by the diagrams of perturbation theory above some effective threshold s_{eff} . Thus, the only — although really essential — ingredient of the LD sum rule for the pion elastic form factor is this effective continuum threshold $s_{\text{eff}}(Q^2)$. Let us emphasize that, since the $O(1)$ and $O(\alpha_s)$ contributions to the pion form factor are governed by one and the same effective threshold $s_{\text{eff}}(Q^2)$, the relative weights of these contributions may be *predicted*. Their ratio $F_\pi^{(0)}(Q^2)/F_\pi^{(1)}(Q^2)$ turns out to be relatively stable with respect to s_{eff} and may therefore be calculated relatively accurately (see Fig. 4).

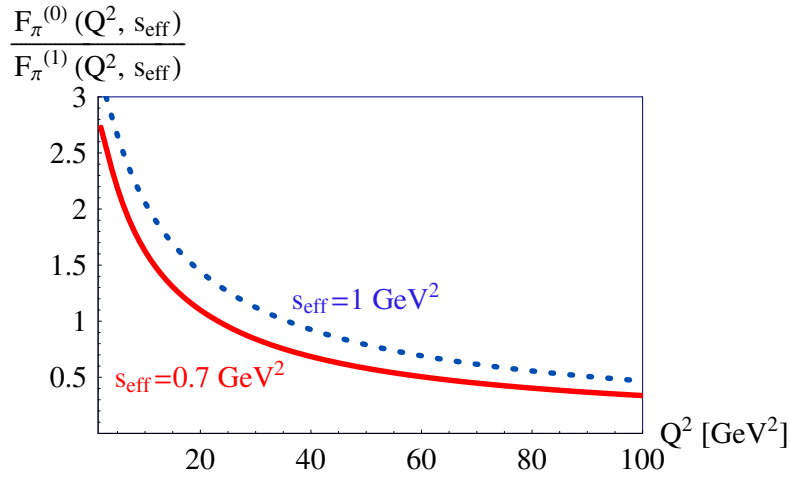


Fig. 4: Ratio $F_\pi^{(0)}(Q^2)/F_\pi^{(1)}(Q^2)$ of $O(1)$ and $O(\alpha_s)$ contributions to the pion elastic form factor vs. Q^2 for two reasonable values of the — here by assumption constant — effective threshold s_{eff} .

Quark–hadron duality implies that the effective threshold (although being a function of Q^2) always — that is, also for large Q^2 — stays in a region close to 1 GeV². Moreover, in order to satisfy the QCD factorization theorem for the $O(\alpha_s)$ contribution to the form factor, the effective threshold should behave like $s_{\text{eff}}(Q^2) \rightarrow 4\pi^2 f_\pi^2$ for $Q^2 \rightarrow \infty$. This requirement has immediate consequences for the large- Q^2 behaviour of different contributions to the pion form factor:

1. Since $s_{\text{eff}}(Q^2)$ is bounded from above, the $O(1)$ contribution $F_\pi^{(0)}(Q^2)$ to the elastic form factor $F_\pi(Q^2)$ of the pion behaves like $F_\pi^{(0)}(Q^2) \propto 1/Q^4$ for $Q^2 \rightarrow \infty$. We would like to emphasize that the decrease of the soft contribution to the pion elastic form factor like $1/Q^4$ is a direct consequence of perturbation theory and quark–hadron duality.
2. Consequently, for large Q^2 the pion elastic form factor $F_\pi(Q^2)$ is dominated by the $O(\alpha_s)$ contribution $F_\pi^{(1)}(Q^2)$.

Now, in the holographic models of [1, 2] merely the soft contribution is considered: it behaves like $1/Q^2$ for large Q^2 . This is only possible if the effective threshold $s_{\text{eff}}(Q^2)$ rises with Q^2 . However, this immediately leads to the violation of the factorization theorem for the $O(\alpha_s)$ contribution, which is governed by the same effective threshold. Consequently, we would like to emphasize that the findings of [1, 2] would imply that the QCD factorization theorem is violated. We thus conclude that the predictions of [1, 2] for the pion form factor seem to us improbable.⁵

Needless to say, the *conventional* LD model [16] for the effective continuum threshold $s_{\text{eff}}(Q^2)$, defined by the choice $s_{\text{eff}}(Q^2) = 4\pi^2 f_\pi^2$ for all Q^2 , or the slightly more sophisticated approach of [4] are approximations which do not account for all the subtle details of the confinement dynamics. Consequently, it is of utmost importance to acquire a satisfactory understanding of the accuracy to be expected within this approach, in other words, to obtain some reliable estimate of the expected deviations of the exact $s_{\text{eff}}(Q^2)$ from its LD limit s_{LD} in the momentum region $Q^2 \geq 4\text{--}6$ GeV². To this end, let us take advantage of the fact that in quantum mechanics all the bound-state features may be found exactly by solving the Schrödinger equation. On the other hand, also within quantum mechanics we may construct LD sum rules.

4. LOCAL-DUALITY MODEL FOR THE ELASTIC FORM FACTOR IN QUANTUM MECHANICS

Factorization of hard form factors for large momentum transfers is the main relevant ingredient of the approach to the pion form factor advocated in [4]. Hence, for any interaction of Coulomb-plus-confining type this model can be tested in quantum mechanics, which has already proven to be a rather efficient tool for studying various features of QCD [21–25].

Within potential models, the elastic form factor of the ground state, $F(Q)$, is given in terms of the wave function Ψ by

$$F(Q) = \int d^3r \exp(i\mathbf{q} \cdot \mathbf{r}) |\Psi(\mathbf{r})|^2 = \int d^3k \Psi(\mathbf{k}) \Psi(\mathbf{k} + \mathbf{q}), \quad Q \equiv |\mathbf{q}|, \quad (4.1)$$

where the bound-state wave function Ψ is computed by solving the stationary Schrödinger equation for the Hamiltonian

$$H = \frac{\mathbf{k}^2}{2m} - \frac{\alpha}{r} + V_{\text{conf}}(r), \quad r \equiv |\mathbf{r}|. \quad (4.2)$$

At large values of Q , the asymptotic behaviour of the elastic form factor $F(Q)$ is given by the factorization theorem [26]:

$$F(Q) \xrightarrow{Q \rightarrow \infty} F_\infty(Q) \equiv \frac{16\pi \alpha m R_g}{Q^4}, \quad R_g \equiv |\Psi(\mathbf{r} = \mathbf{0})|^2. \quad (4.3)$$

The quantum-mechanical LD sum rules for decay constant R_g and form factor $F(Q)$ are rather similar to those in QCD:

$$R_g = \int_0^{\bar{k}_{\text{eff}}} dk \rho_{\text{pert}}^{\text{QM}}(k), \quad (4.4)$$

$$F_g(Q) R_g = \int_0^{k_{\text{eff}}(Q)} \int_0^{k_{\text{eff}}(Q)} dk_1 dk_2 \Delta_{\text{pert}}^{\text{QM}}(k_1, k_2, Q). \quad (4.5)$$

The spectral densities, $\rho_{\text{pert}}^{\text{QM}}$ and $\Delta_{\text{pert}}^{\text{QM}}$, are calculated from two- and three-point diagrams of nonrelativistic field theory; the derivation of these expressions up to $O(\alpha_s)$ accuracy is presented in Appendix A.

⁵ A way out would be to assume different effective thresholds for the $O(1)$ and the $O(\alpha_s)$ contributions to the pion form factor. This seems, however, a rather artificial construction.

The factorization limit (4.3) means for the momentum-dependent effective threshold in the three-point sum rule (4.5)

$$k_{\text{eff}}(Q) \xrightarrow{Q \rightarrow \infty} k_{\text{LD}} \equiv (6\pi^2 R_g)^{1/3}. \quad (4.6)$$

For intermediate Q , the behaviour of $F(Q)$ is controlled by the details of the confining interaction via the corresponding wave function Ψ . The LD model assumes that, also for intermediate Q , one may find a reasonable estimate for the form factor by setting $k_{\text{eff}}(Q) = k_{\text{LD}}$. Hence, similar to QCD the only property of the bound state which determines the form factor in the LD framework is R_g . To probe the accuracy of this approximation, we consider a set of confining potentials

$$V_{\text{conf}}(r) = \sigma_n (m r)^n, \quad n = 2, 1, 1/2. \quad (4.7)$$

We adopt parameter values appropriate for hadron physics, i.e., $m = 0.175$ GeV for the reduced constituent light-quark mass and $\alpha = 0.3$, and adapt the strengths σ_n in our confining interactions such that the Schrödinger equation yields for each potential the same $\Psi(\mathbf{r} = \mathbf{0}) = 0.078 \text{ GeV}^{3/2}$, which holds for $\sigma_2 = 0.71 \text{ GeV}$, $\sigma_1 = 0.96 \text{ GeV}$, and $\sigma_{1/2} = 1.4 \text{ GeV}$.

Figure 5 summarizes our findings in quantum mechanics. In the region $Q < 2 \text{ GeV}$, the exact $k_{\text{eff}}(Q)$ is a complicated function of Q (Fig. 5b). Moreover, its behaviour exhibits features similar to the exact QCD threshold, recalculated from pion form-factor data (see Fig. 3). In the region $Q > 2 \text{ GeV}$, the exact $k_{\text{eff}}(Q)$ is smoothly approaching its LD limit k_{LD} . For a steeply rising confining potential ($n = 2$), the LD model works with, e.g., 5% accuracy already for $Q > 2 \text{ GeV}$. For a slowly rising confining potential ($n = 1/2$), such high accuracy is reached not before $Q \approx 6 \text{ GeV}$. However, we identify an important universal feature that does not depend on any details of the confining interaction: The accuracy of the LD approximation for the effective threshold, $k_{\text{eff}}(Q) \approx k_{\text{LD}}$, as well as the accuracy of the corresponding elastic form factor increase with Q in the region $Q^2 \geq 4 \text{ GeV}^2$. Figure 5 illustrates this observation for several confining potentials with different large- r behaviour. On the basis of these findings we are forced to conclude that, if at $Q^2 \approx 4\text{--}8 \text{ GeV}^2$ the LD model provides a good description of the data, the accuracy of this model won't become worse at larger values of Q^2 .

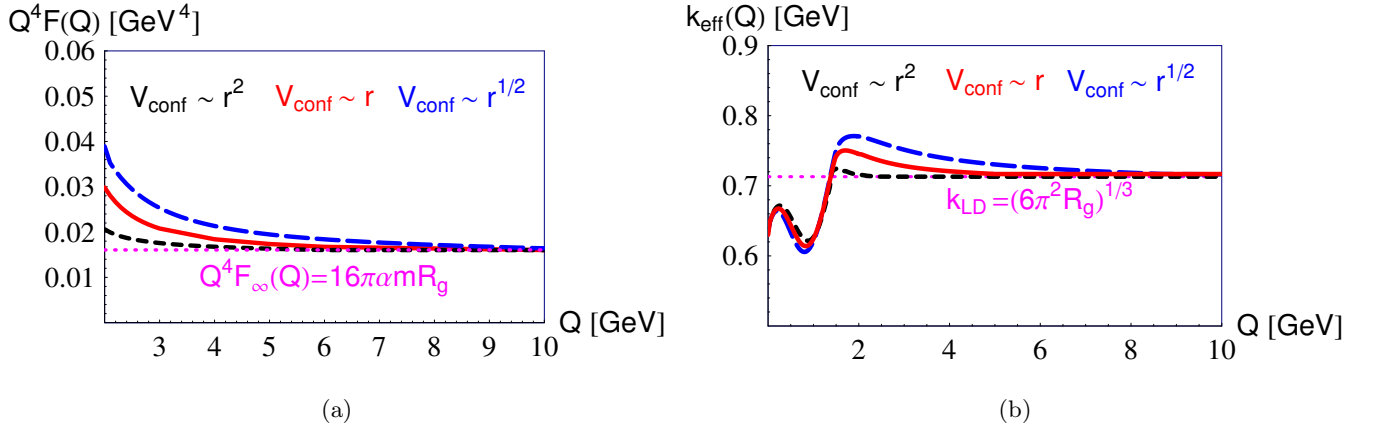


Fig. 5: (a) Exact vs. LD ground-state form factors $F(Q)$. (b) Exact vs. LD effective thresholds $k_{\text{eff}}(Q)$. Dotted (magenta) lines: local-duality limit; short-dashed (black) lines: harmonic-oscillator confinement, $n = 2$; full (red) lines: linear confinement, $n = 1$; long-dashed (blue) lines: square-root confinement, $n = 1/2$.

The crucial question is to which extent these findings can be carried over to predictions for the pion elastic form factor by LD sum rules in QCD. In spite of subtleties related to evident differences between the spectral densities in relativistic and nonrelativistic theories [27], we are convinced that the pivotal lesson from quantum mechanics also applies to QCD: Above the relatively low value $Q^2 \approx 4\text{--}6 \text{ GeV}^2$, the accuracy of the approximation $s_{\text{eff}}(Q^2) \approx s_{\text{LD}}$ will *increase* with Q^2 . The results for the form factor found in [1–3, 5] clearly contradict this observation: the equivalent effective threshold for these models deviates from the LD threshold $s_{\text{LD}} \equiv 4\pi^2 f_{\pi}^2$ in the broad range $Q^2 \approx 4\text{--}10 \text{ GeV}^2$ and for [1–3] even up to $Q^2 \approx 100 \text{ GeV}^2$; more importantly, the deviations between the equivalent thresholds and the LD threshold rise with Q^2 .

5. CONCLUSIONS AND OUTLOOK

We investigated the expected accuracy of the pion elastic form factor obtained by the LD version of QCD sum rules. The LD sum rule for the elastic form factor of some bound state may be constructed in any theory where hard exclusive

amplitudes satisfy the factorization theorem (that is, in essence, in any theory with interactions behaving Coulomb-like at small distances and confining at large distances). In this approach, the form factor is determined by two ingredients: the double spectral densities of the diagrams of perturbation theory and the Q^2 -dependent effective threshold $s_{\text{eff}}(Q^2)$. The effective threshold satisfies rigorous constraints at $Q^2 = 0$ and for $Q^2 \rightarrow \infty$ but is unknown for intermediate values of Q^2 . The LD model assumes that approximating $s_{\text{eff}}(Q^2)$ by its value at $Q^2 \rightarrow \infty$, $4\pi^2 f_\pi^2$, yields reasonable estimates for the form factor at not too small Q^2 . We have tested the accuracy of the LD model in quantum mechanics, where the exact effective threshold may be calculated from the exact form factor, obtained by solving the Schrödinger equation.

The new results reported in this work are the following:

1. For $Q^2 \leq 4 \text{ GeV}^2$, the exact effective threshold exhibits a rapid variation with Q^2 , implying that in this region the accuracy of the LD model depends on the details of the confining interactions and cannot be predicted in advance.
2. For $Q^2 \geq 4\text{--}6 \text{ GeV}^2$, independently of the details of the confining interactions, the *maximal* deviation of the exact effective threshold from the LD approximation is reached for $Q^2 \approx 4\text{--}6 \text{ GeV}^2$. As Q^2 increases further, the exact effective threshold approaches the LD limit quickly, thus improving the accuracy of the LD model for the elastic form factor. For generic confining interactions, the LD approach gives very precise results for $Q^2 \geq 20\text{--}30 \text{ GeV}^2$. In other words, it is not possible to predict at which value Q_*^2 of the momentum transfer the LD model provides a good approximation (with, say, 5% accuracy) to the exact form factor, as the precise Q_*^2 depends on subtle details of the confining interactions. However, one important conclusion may be drawn from our quantum-mechanical analysis: *independently* of any subtle details, the accuracy of the LD approximation *increases* for $Q^2 \geq 4\text{--}6 \text{ GeV}^2$.
3. Existing data on the pion elastic form factor indicate that the LD limit $s_{\text{LD}} \equiv 4\pi^2 f_\pi^2$ of the effective threshold may be reached already at the relatively low values $Q^2 \approx 5\text{--}6 \text{ GeV}^2$ of the momentum transfer.

On the basis of the results obtained in this study, we are forced to conclude that those large deviations from the LD limit in the range $Q^2 = 20\text{--}50 \text{ GeV}^2$ reported in [1, 2] appear to us highly unlikely.

Of course, our analysis does not provide a proof of but an argument for the accuracy of the LD approximation in QCD and it gives some hint towards the behaviour of the pion elastic form factor $F_\pi(Q^2)$ to be expected for large values of Q^2 . Accordingly, the accurate experimental determination of $F_\pi(Q^2)$ for $Q^2 = 4\text{--}10 \text{ GeV}^2$ expected to be achieved by JLab will have important implications for the predicted behaviour of $F_\pi(Q^2)$ at large Q^2 up to asymptotically high momenta.

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Appendix A: Perturbative expansion of Green functions in quantum mechanics

We construct the perturbative expansions of both polarization operator and vertex function in quantum mechanics.

1. Polarization operator

The polarization operator $\Pi(E)$ is defined by [23]

$$\Pi(E) = \langle \mathbf{r}' = \mathbf{0} | G(E) | \mathbf{r} = \mathbf{0} \rangle, \quad (\text{A.1})$$

where $G(E)$ is the full Green function, i.e., $G(E) = (H - E)^{-1}$, defined by the model Hamiltonian under consideration

$$H = H_0 + V(r), \quad H_0 \equiv \frac{\mathbf{k}^2}{2m}, \quad r \equiv |\mathbf{r}|. \quad (\text{A.2})$$

The expansion of the full Green function $G(E)$ in powers of the interaction potential V has the well-known form

$$G(E) = G_0(E) - G_0(E)V G_0(E) + G_0(E)V G_0(E)V G_0(E) + \dots, \quad (\text{A.3})$$

with $G_0(E) = (H_0 - E)^{-1}$. It generates the corresponding expansion of $\Pi(E)$:

$$\Pi(E) = \Pi_0(E) + \Pi_1(E) + \dots. \quad (\text{A.4})$$

Explicitly, one finds

$$\Pi_0(E) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\frac{k^2}{2m} - E}, \quad (\text{A.5})$$

$$\Pi_1(E) = -\frac{1}{(2\pi)^6} \int \frac{d^3k}{\frac{k^2}{2m} - E} \frac{d^3k'}{\frac{k'^2}{2m} - E} V((\mathbf{k} - \mathbf{k}')^2). \quad (\text{A.6})$$

We consider interaction potentials $V(r)$ which consist of a Coulombic and a confining part:

$$V(r) = -\frac{\alpha}{r} + V_{\text{conf}}(r). \quad (\text{A.7})$$

Then the expansion (A.4) becomes a double expansion in powers of the Coulomb coupling α and the confining potential V_{conf} (see Fig. 6).

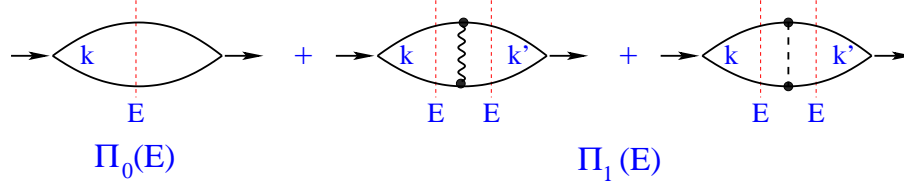


Fig. 6: Expansion of the polarization operator in terms of Coulomb (wavy line) and confining (dashed line) interaction potentials.

The contribution to $\Pi(E)$ arising from the Coulombic potential is referred to as the perturbative contribution, Π_{pert} . The contributions involving the confining potential V_{conf} (including the mixed terms receiving contributions from both confining and Coulomb parts) are referred to as the power corrections, Π_{power} . For instance, the first-order perturbative contribution reads

$$\Pi_1^{(\alpha)}(E) = \frac{1}{(2\pi)^6} \int \frac{d^3k}{\frac{k^2}{2m} - E} \frac{d^3k'}{\frac{k'^2}{2m} - E} \frac{4\pi\alpha}{(\mathbf{k} - \mathbf{k}')^2} = \frac{\alpha m}{8\pi^2} \int \frac{d^3k}{(\frac{k^2}{2m} - E) |\mathbf{k}|}. \quad (\text{A.8})$$

The integral diverges but becomes convergent after applying the Borel transformation $1/(a - E) \rightarrow \exp(-aT)$.⁶ The Borel-transformed polarization operator $\Pi(T)$ has the form

$$\Pi(T) = \Pi_{\text{pert}}(T) + \Pi_{\text{power}}(T), \quad \Pi_{\text{pert}}(T) = \left(\frac{m}{2\pi T}\right)^{3/2} \left[1 + \sqrt{2\pi m T} \alpha + \frac{1}{3} m \pi^2 T \alpha^2 + O(\alpha^3)\right]. \quad (\text{A.9})$$

In the LD limit, that is, for $T \rightarrow 0$, only $\Pi_{\text{pert}}(T)$ will be relevant. Nevertheless, as an illustration we provide also the result for the power corrections $\Pi_{\text{power}}(T)$ for the case of a harmonic-oscillator confining potential $V_{\text{conf}}(r) = m\omega^2 r^2/2$:

$$\Pi_{\text{power}}(T) = \left(\frac{m}{2\pi T}\right)^{3/2} \left[-\frac{1}{4} \omega^2 T^2 \left(1 + \frac{11}{12} \sqrt{2\pi m T} \alpha\right) + \frac{19}{480} \omega^4 T^4\right]. \quad (\text{A.10})$$

Let us point out that $\Pi_{\text{power}}(T = 0)$ vanishes, similar to QCD. The radiative corrections in $\Pi_{\text{pert}}(T)$ have a less singular behaviour compared to the free Green function, so the system behaves as quasi-free system. In QCD such a behaviour, frequently regarded as an indication of asymptotic freedom, occurs due to the running of the strong coupling α_s and its vanishing at small distances. Interestingly, in the nonrelativistic potential model this feature is built-in automatically.

Now, according to the standard procedures of the method of sum rules, the dual correlator is obtained by applying a low-energy cut at some threshold \bar{k}_{eff} in the spectral representation for the perturbative contribution to the correlator:

$$\Pi_{\text{dual}}(T, \bar{k}_{\text{eff}}) = \frac{1}{2\pi^2} \int_0^{\bar{k}_{\text{eff}}} dk k^2 \exp\left(-\frac{k^2}{2m} T\right) \left[1 + \frac{\pi m \alpha}{k} + \frac{(\pi m \alpha)^2}{3k^2} + O(\alpha^3)\right] + \Pi_{\text{power}}(T). \quad (\text{A.11})$$

⁶ Note that the Borel transform of the Green function $(H - E)^{-1}$ yields the quantum-mechanical time-evolution operator in imaginary time $U(T) = \exp(-HT)$.

By construction, the dual correlator $\Pi_{\text{dual}}(T, \bar{k}_{\text{eff}})$ is related to the ground-state contribution by

$$\Pi_{\text{dual}}(T, \bar{k}_{\text{eff}}) = \Pi_{\text{g}}(T) \equiv R_{\text{g}} \exp(-E_{\text{g}}T), \quad R_{\text{g}} \equiv |\psi_{\text{g}}(r=0)|^2. \quad (\text{A.12})$$

As we have shown in our previous studies of potential models, the effective continuum threshold defined according to (A.12) is a function of the Borel time parameter T . For $T = 0$, one finds

$$\Pi_{\text{dual}}(\bar{k}_{\text{eff}}, T = 0) = \frac{1}{6\pi^2} \bar{k}_{\text{eff}}^3 + \frac{\alpha m}{4\pi} \bar{k}_{\text{eff}}^2 + \dots. \quad (\text{A.13})$$

2. Vertex function

We now calculate the vertex function $\Gamma(E, E', Q)$, defined by

$$\Gamma(E, E', Q) = \langle \mathbf{r}' = \mathbf{0} | G(E) J(\mathbf{q}) G(E') | \mathbf{r} = \mathbf{0} \rangle, \quad Q \equiv |\mathbf{q}|, \quad (\text{A.14})$$

where $J(\mathbf{q})$ is the operator which adds a momentum \mathbf{q} to the interacting constituent. The expansions (A.3) of the full Green functions $G(E)$ and $G(E')$ in powers of the interaction entail a corresponding expansion of $\Gamma(E, E', Q)$, cf. Fig. 7:

$$\Gamma(E, E', Q) = \Gamma_0(E, E', Q) + \Gamma_1(E, E', Q) + \dots. \quad (\text{A.15})$$

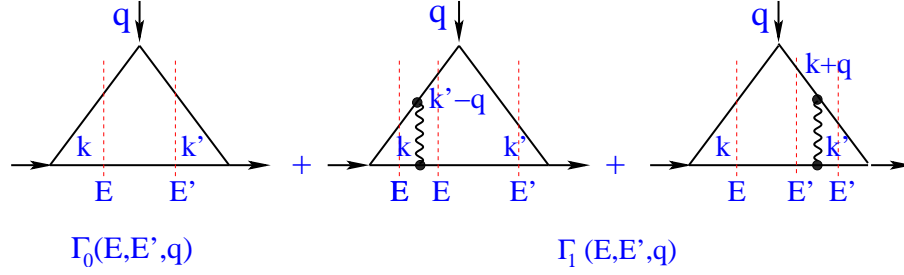


Fig. 7: Nonrelativistic Feynman diagrams representing the lowest perturbative contributions to the vertex function $\Gamma(E, E', Q)$.

For the vertex functions $\Gamma_i(E, E', Q)$, $i = 0, 1, \dots$, in (A.15), their double spectral representations may be written as

$$\Gamma_i(E, E', Q) = \int \frac{dz}{\frac{z}{2m} - E} \frac{dz'}{\frac{z'}{2m} - E'} \Delta_i(z, z', Q). \quad (\text{A.16})$$

The vertex functions $\Gamma_i(E, E', Q = 0)$ and the polarization operators $\Pi_i(E)$ satisfy the Ward identities

$$\Gamma_i(E, E', Q = 0) = \frac{\Pi_i(E) - \Pi_i(E')}{E - E'}, \quad (\text{A.17})$$

which are equivalent to the following relations between the corresponding spectral densities:

$$\lim_{Q \rightarrow 0} \Delta_i(z, z', Q) = \delta(z - z') \rho_i(z). \quad (\text{A.18})$$

We shall consider the double Borel transform $E \rightarrow T$ and $E' \rightarrow T'$: $(a - E)^{-1} \rightarrow \exp(-aT)$, $(a' - E')^{-1} \rightarrow \exp(-a'T')$. Equation (A.18) leads to the following Ward identities for the Borel images:

$$\Gamma_i(T, T', Q = 0) = \Pi_i(T + T'). \quad (\text{A.19})$$

a. One-loop contribution Γ_0 to the vertex function

The zero-order one-loop term has the form

$$\Gamma_0(E, E', Q) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\left(\frac{\mathbf{k}^2}{2m} - E\right) \left(\frac{(\mathbf{k} + \mathbf{q})^2}{2m} - E'\right)}. \quad (\text{A.20})$$

This may be written as the double spectral representation

$$\Gamma_0(E, E', Q) = \int \frac{dz}{\frac{z}{2m} - E} \frac{dz'}{\frac{z'}{2m} - E'} \Delta_0(z, z', Q), \quad (\text{A.21})$$

where

$$\Delta_0(z, z', Q) = \frac{1}{(2\pi)^3} \int d^3k \delta(z - \mathbf{k}^2) \delta(z' - (\mathbf{k} - \mathbf{q})^2) = \frac{1}{(2\pi)^3} \frac{\pi}{2Q} \theta((z' - z - Q^2)^2 - 4zQ^2 < 0). \quad (\text{A.22})$$

Hereafter, we use the notations $k \equiv \sqrt{z}$ and $k' \equiv \sqrt{z'}$. In terms of the variables k and k' , the θ function takes the form

$$\theta((z' - z - Q^2)^2 - 4zQ^2 < 0) = \theta(|k - Q| < k' < k + Q). \quad (\text{A.23})$$

b. Two-loop contribution Γ_1 to the vertex function

We consider here only corrections related to the *Coulomb* potential, since power corrections induced by the confining interaction vanish in the LD limit. The two-loop $O(\alpha)$ correction receives two contributions and has the form

$$\Gamma_1(E, E', Q) = \frac{1}{(2\pi)^6} \int \frac{d^3k}{\frac{\mathbf{k}^2}{2m} - E} \frac{d^3k'}{\frac{\mathbf{k}'^2}{2m} - E'} \frac{4\pi\alpha}{(\mathbf{k} - (\mathbf{k}' - \mathbf{q}))^2} \left[\frac{1}{\frac{(\mathbf{k}' - \mathbf{q})^2}{2m} - E'} + \frac{1}{\frac{(\mathbf{k} + \mathbf{q})^2}{2m} - E} \right]. \quad (\text{A.24})$$

Having in mind the subsequent application of a double Borel transformation in E and E' , it is convenient to represent Γ_1 as a sum of two terms, $\Gamma_1 = \Gamma_1^{(a)} + \Gamma_1^{(b)}$, with

$$\begin{aligned} \Gamma_1^{(a)}(E, E', Q) &= \frac{\alpha m}{8\pi^5} \int \frac{d^3k'}{(\frac{\mathbf{k}'^2}{2m} - E') \left(\frac{(\mathbf{k}' - \mathbf{q})^2}{2m} - E \right)} \int \frac{d^3k}{(\mathbf{k} - (\mathbf{k}' - \mathbf{q}))^2 [\mathbf{k}^2 - (\mathbf{k}' - \mathbf{q})^2]} \\ &\quad + \frac{\alpha m}{8\pi^5} \int \frac{d^3k}{(\frac{\mathbf{k}^2}{2m} - E) \left(\frac{(\mathbf{k} + \mathbf{q})^2}{2m} - E' \right)} \int \frac{d^3k'}{((\mathbf{k} + \mathbf{q}) - \mathbf{k}')^2 [\mathbf{k}'^2 - (\mathbf{k} + \mathbf{q})^2]}, \\ \Gamma_1^{(b)}(E, E', Q) &= \frac{\alpha m}{8\pi^5} \int \frac{d^3k}{\frac{\mathbf{k}^2}{2m} - E} \frac{d^3k'}{\frac{\mathbf{k}'^2}{2m} - E'} \frac{1}{(\mathbf{k} + \mathbf{q} - \mathbf{k}')^2} \left[\frac{1}{(\mathbf{k}' - \mathbf{q})^2 - \mathbf{k}^2} + \frac{1}{(\mathbf{k} + \mathbf{q})^2 - \mathbf{k}'^2} \right]. \end{aligned} \quad (\text{A.25})$$

The double Borel transformation in $E \rightarrow T$ and $E' \rightarrow T'$ is now easily performed.

Let us start with $\Gamma_1^{(a)}$. One integration in $\Gamma_1^{(a)}$ may be performed, leading to

$$\Gamma_1^{(a)}(E, E', Q) = \frac{\alpha m}{16\pi^2} \left[\int \frac{d^3k'}{\left(\frac{(\mathbf{k}' + \mathbf{q})^2}{2m} - E' \right) \left(\frac{\mathbf{k}'^2}{2m} - E \right) |\mathbf{k}'|} + \int \frac{d^3k}{\left(\frac{(\mathbf{k} - \mathbf{q})^2}{2m} - E \right) \left(\frac{\mathbf{k}^2}{2m} - E' \right) |\mathbf{k}|} \right]. \quad (\text{A.26})$$

The first term corresponds to the contribution of the “left” two-loop diagram in Fig. 7, i.e., with the potential before the interaction with the current $J(\mathbf{q})$, while the second term is represented by the “right” two-loop diagram in Fig. 7. The corresponding double spectral densities have a form very similar to Δ_0 :

$$\Delta_{1L}^{(a)}(k, k', Q) = \frac{\alpha m}{16\pi} \frac{\pi}{2Qk} \theta(|k - Q| < k' < k + Q) \theta(0 < k) \theta(0 < k'), \quad \Delta_{1R}^{(a)}(k, k', Q) = \Delta_{1L}^{(a)}(k', k, Q). \quad (\text{A.27})$$

Explicit calculations yield the following double spectral densities of the two contributions to $\Gamma_1^{(b)}$ related to the “left” and “right” two-loop diagrams in Fig. 7:

$$\Delta_{1L}^{(b)}(k, k', Q) = \frac{\alpha m}{32\pi^6} \frac{1}{Qk} \left[\log^2 \left(\left| \frac{k' - Q + k}{k' - Q - k} \right| \right) - \log^2 \left(\left| \frac{k' + Q + k}{k' + Q - k} \right| \right) \right], \quad \Delta_{1R}^{(b)}(k, k', Q) = \Delta_{1L}^{(b)}(k', k, Q). \quad (\text{A.28})$$

At $Q = 0$, $\Gamma_1^{(a)}(T, T', Q = 0)$ satisfies the Ward identity, $\Gamma_1^{(a)}(T, T', Q = 0) = \Pi_1^{(\alpha)}(T + T')$, whereas $\Gamma_1^{(b)}(T, T', Q = 0)$ vanishes: $\Gamma_1^{(b)}(T, T', Q = 0) = 0$. For large Q and $T, T' \neq 0$, $\Gamma_1^{(b)}(T, T', Q)$ assumes a factorizable form (see Eq. (A.25)):

$$\Gamma_1^{(b)}(T, T', Q) \rightarrow \frac{16\pi\alpha m}{Q^4} \Pi_0(T) \Pi_0(T'). \quad (\text{A.29})$$

At the same time, both $\Gamma_0(T, T', Q)$ and $\Gamma_1^{(a)}(T, T', Q)$ are exponentially suppressed for large Q and $T, T' \neq 0$. Hence, $\Gamma_1^{(b)}$ determines the large- Q behaviour of the vertex function.

c. Dual correlator

The dual correlator $\Gamma_{\text{dual}}(T, T', Q)$ is constructed in a standard way, by application of a low-energy cut to the double spectral representation of the perturbative contribution to (A.16):

$$\Gamma_{\text{dual}}(T, T', Q) = \int_0^{k_{\text{eff}}(Q, T)} dk \, 2k \exp\left(-\frac{k^2}{2m} T\right) \int_0^{k_{\text{eff}}(Q, T')} dk' \, 2k' \exp\left(-\frac{k'^2}{2m} T'\right) \Delta(z, z', Q) + \Gamma_{\text{power}}(T, T', Q). \quad (\text{A.30})$$

By construction, the dual correlator corresponds to the ground-state contribution $\exp(-E_g T) \exp(-E_g T') R_g F_g(Q)$.

In the LD limit $T = 0$ and $T' = 0$, $\Gamma_{\text{power}}(T, T', Q)$ vanishes and the ground-state form factor $F_g(Q)$ is related to the low-energy part of the perturbative contribution considered above:

$$\int_0^{k_{\text{eff}}(Q)} dk \, 2k \int_0^{k_{\text{eff}}(Q)} dk' \, 2k' \Delta(k, k', Q) = F_g(Q) R_g, \quad (\text{A.31})$$

with $\Delta(k, k', Q) = \Delta_0(k, k', Q) + \Delta_{1L}^{(a)}(k, k', Q) + \Delta_{1R}^{(a)}(k, k', Q) + \Delta_{1L}^{(b)}(k, k', Q) + \Delta_{1R}^{(b)}(k, k', Q) + O(\alpha^2)$.

In order to provide the correct normalization $F_g(Q = 0) = 1$ of the elastic form factor $F_g(Q)$, the effective thresholds should be related to each other according to $k_{\text{eff}}(Q = 0) = \bar{k}_{\text{eff}}$; then the form factor is correctly normalized due to the Ward identity (A.18) satisfied by the spectral densities.

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